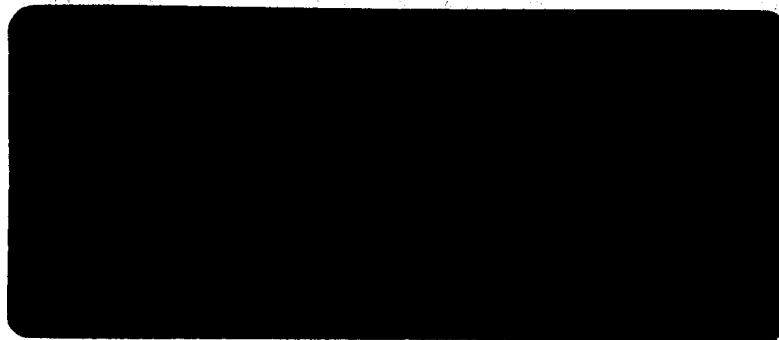


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**THE ASYMPTOTIC THEORY OF CONTROL SYSTEMS;  
I. STOCHASTIC AND DETERMINISTIC PROCESSES**

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## THE ASYMPTOTIC THEORY OF CONTROL SYSTEMS; 1. STOCHASTIC AND DETERMINISTIC PROCESSES<sup>1</sup>

David S. Adorno

In this paper we study the asymptotic behavior of a particular class of control systems, with particular emphasis being placed on the questions of convergence and the steady-state forms of the loss function and of the system itself. The concept of policy equivalence is introduced and used to show that a stochastic, linear, first-order control system with quadratic loss function is policy equivalent to the deterministic system which depends only on the expected states of the system. With this theorem as justification, only deterministic systems are considered for the remainder of the paper. The extension of this topic to the adaptive control case (the case in which the parameters of the random noise are not known) is currently under study and will be reported on in a forthcoming paper. For an appreciation of the adaptive control case, the reader is referred to Bellman [1 and 2] and Freimer [3 and 4]. The particular class of control systems referred to above is the class of linear, first-order, matrix difference equations with quadratic loss criterion.

For an  $N$ -stage process in this class, the technique of dynamic programming is used to determine the optimal policy. From this technique, we obtain the forms of the loss function and the state vector under optimal control. Convergence of the loss function is established by showing its monotonic behavior together with uniform boundedness. At this point, it is noted that the loss function under optimal policy, the optimal policy, and the control system under optimal policy, are all functions of a certain sequence of matrices  $R_N$ . Hence, the characterization of the asymptotic properties of the system all depend on the asymptotic properties of the sequence  $R_N$ . This sequence is highly non-linear and very difficult to study in this form. Therefore, a technique is introduced to convert the non-linear sequence into a set of two simultaneous linear difference equations. The behavior of this system is then studied, and examples are given.

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## I. INTRODUCTION

Consider the  $N$ -stage control system (hereafter referred to as the system  $S$ ) defined by the linear-vector stochastic difference equation

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n + \mathbf{Y}_n + \mathbf{R}_n, \quad \mathbf{X}_1 = \mathbf{C}, \quad n = 1, 2, 3, \dots, N \quad (1)$$

(Bold-face type represents vectors or matrices.)

The vectors in Eq. 1 have the following interpretation:

1.  $\mathbf{X}_n$  is a  $p \times 1$  vector which denotes the state of the system at step  $n$ . That is, the system is describable by  $p$  components, and the  $i$ th component  $X_{i,n}$  of  $\mathbf{X}_n$ ,  $i = 1, 2, \dots, p$  denotes the state of the  $i$ th component of  $S$  on step  $n$ .
2.  $\mathbf{A}$  is a  $p \times p$  constant matrix, called the system transformation.
3.  $\mathbf{Y}_n$  is a  $p \times 1$  vector which denotes the amount of control that will be forced on  $S$  at step  $n$ . The choice of  $N$  control vectors  $\mathbf{Y}_1, \mathbf{Y}_2, \dots, \mathbf{Y}_N$ , for  $S$  symbolized by  $\mathbf{Y}$ , is called a *policy*.
4.  $\mathbf{R}_n$  is a sequence of independent, identically distributed random vectors. The statement that  $\mathbf{R}_n$  is a random vector means that  $\mathbf{R}_n$  is a  $p \times 1$  vector and the components  $R_{in}$ ,  $i = 1, \dots, p$  are random variables with a joint distribution.
5.  $\mathbf{C}$  is a  $p \times 1$  vector which denotes the initial state of the system.

If the system  $S$  were allowed to run without control (or equivalently  $\mathbf{Y}_i = \mathbf{0}$ ,  $i = 1, 2, \dots, N$ ), then Eq. 1 would reduce to

$$\mathbf{X}_{n+1} = \mathbf{A}\mathbf{X}_n + \mathbf{R}_n, \quad \mathbf{X}_1 = \mathbf{C}, \quad n = 1, \dots, N \quad (2)$$

It is desired to control  $S$  so that  $\mathbf{X}_n = \mathbf{0}$ , and, we measure the cost of deviation from this zero-state at the  $n$ th step by  $\mathbf{X}_n' \mathbf{X}_n$ , where the prime denotes transposition. Likewise, the cost of control at the  $n$ th step is  $\delta \mathbf{Y}_n' \mathbf{Y}_n$ ,  $\delta > 0$ . Then, for a particular policy  $\mathbf{Y}$ , the expected cost of operating this  $N$ -stage control system  $S$  becomes

$$L_N(\mathbf{C}, \mathbf{Y}) = \sum_{n=1}^N E[\mathbf{X}_n' \mathbf{X}_n] + \delta \sum_{n=1}^N \mathbf{Y}_n' \mathbf{Y}_n \quad (3)$$

where  $X_n$  is defined by Eq. 1 and  $E$  denotes mathematical expectation. We shall use this quadratic form as the loss criterion for  $S$ .

*Definition 1:* The statement that  $Y^*$  is an *optimal* policy for  $S$  means that

$$L_N(C, Y^*) = \min_Y [L_N(C, Y)]$$

We denote  $L_N(C, Y^*)$  by  $L_N^*(C)$ .

*Definition 2:* The statement that a system  $S_1$  is *policy equivalent* to the system  $S$  means that if  $Y^*$  is an optimal policy in  $S_1$ , then  $Y^*$  is also an optimal policy in  $S$ , and conversely.

*Definition 3:* The statement that  $S$  is *deterministic* means that  $R_n = 0$  with probability one,  $n = 1, 2, 3, \dots$ .

### III. STOCHASTIC SYSTEMS

Before we can prove the policy-equivalence theorem stated in the summary, a lemma will be needed.

*Lemma 1.* Let  $X_n$  be defined by Eq. 1. If  $E(R_n)$  and  $\sigma^2(R_n)$  both exist, then

$$\sigma^2(X_{n+1}) = \text{trace}(A' A M_n) + \sigma^2(R_n)$$

where  $M_n$  is the covariance matrix of the coordinates of  $X_n$ .

*Proof:* Let  $n \geq 2$ ,  $X_n = AX_{n-1} + Y_{n-1} + R_{n-1}$ ,  $Z_n = X_n - EX_n$ , and  $W_n = R_n - ER_n$ ; then  $E(Z_n) = 0 = E(W_n)$ ,  $E(Z_n' Z_n) = \sigma^2(X_n)$ .

$$\begin{aligned} \sigma^2(X_n) &= E[(AZ_{n-1} + W_{n-1})'(AZ_{n-1} + W_{n-1})] \\ &= E[Z_{n-1}' A' A Z_{n-1} + 2W_{n-1}' A Z_{n-1} + W_{n-1}' W_{n-1}] \\ &= E[Z_{n-1}' A' A Z_{n-1}] + \sigma^2(R_{n-1}) \end{aligned}$$

The last identity follows from the fact that  $X_{n-1}$  does not depend on  $R_{n-1}$  but only on  $R_{n-2}, \dots, R_1$ . Since the  $R_n$  are assumed to be independently distributed,  $E(W_{n-1}' A Z_{n-1}) = E(W_{n-1}') A E(Z_{n-1}) = 0$ . Now the form of  $E[Z_{n-1}' A' A Z_{n-1}]$  remains to be determined. Let  $A' A = B$ , and note that  $B$  is real and symmetric. In general, if  $Z$  is a random vector and  $B$  symmetric, then  $B = (B_{ij})$ ,  $Z = (Z_1, \dots, Z_p)'$ ,

$$E[Z' B Z] = \sum_{j=1}^p \sum_{i=1}^p B_{ij} E(Z_i Z_j) = \text{trace } B M$$

where  $M$  is the covariance matrix for the coordinates of  $Z$ . Applying this result to the problem at hand, the lemma is established.

**Theorem 1.** The stochastic system  $S$ , defined by Eq. 1 and subject to Eq. 3, is policy-equivalent to the deterministic system defined by

$$EX_{n+1} = AEX_n + Y_n + ER_n, \quad EX_1 = C, \quad n = 1, 2, \dots, N$$

with loss function

$$L_N(C, Y) = \sum_{n=1}^N E(X_n)' EX_n + \delta \sum_{n=1}^N Y_n' Y_n$$

when  $E(R_n)$  and  $\sigma^2(R_n)$  exist.

*Proof:* It is easy to establish the identity  $\sigma^2(X_n) = E(X_n' X_n) - (EX_n)'(EX_n)$ . Substitution into  $L_N(C, Y)$  yields

$$L_N(C, Y) = \sum_{n=1}^N (EX_n)'(EX_n) + \sum_{n=1}^N \sigma^2(X_n) + \delta \sum_{n=1}^N Y_n' Y_n$$

From Lemma 1,  $\sigma^2(X_{n+1}) = \text{trace}(A'AM_n) + \sigma^2(R_n)$ , where  $M_n$  is the covariance matrix of  $X_{1,n}, X_{2,n}, \dots, X_{p,n}$ . Since the covariances are formed from  $X_{i,n} - EX_{i,n}$ , the effect of control is cancelled, so that  $\sigma^2(X_n)$  is free of  $Y_n$ . The assertion now follows.

The significance of Theorem 1 is clear. It points out that the optimal policy of a stochastic system in this class depends only on the expected states, no matter how wildly the system fluctuates about the expected states. With this theorem as justification, the remainder of this paper will be devoted to the deterministic system.

### III. DETERMINISTIC SYSTEM SOLUTIONS

The  $N$ -stage control process that we wish to analyze now is Eq. 1 without noise, i.e.,

$$\left. \begin{aligned} X_{n+1} &= AX_n + Y_n, \quad X_1 = C, \quad n = 1, 2, \dots, N \\ L_N^*(C, A, Y) &= \sum_{n=1}^N X_n' X_n + \delta \sum_{n=1}^N Y_n' Y_n, \quad \delta > 0 \end{aligned} \right\} \quad (4)$$

At this point, it will be helpful to recall Bellman's Principle of Optimality [5], "An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision," and apply it to our problem.

$$L_N^*(C, A) = \min_Y \left[ C' C + \delta Y_1' Y_1 + \sum_{n=2}^N X_n' X_n + \delta \sum_{n=2}^N Y_n' Y_n \right]$$

But

$$\sum_{n=2}^N (X_n' X_n) + \delta \sum_{n=2}^N (Y_n' Y_n) = L_{N-1}^*(X_2, A, Y)$$

Therefore, if we require that the last  $N - 1$  steps be optimal with regard to the new initial position  $X_2 = AC + Y_1$ , we need merely to choose  $Y_1$  properly. Hence,

$$L_N^*(C, A) = \min_{Y_1} \left[ C' C + \delta Y_1' Y_1 + L_{N-1}^*(AC + Y_1, A) \right] \quad (5)$$

The required loss function is now the solution to this functional equation. To solve Eq. 5, note that  $L_0^*(C, A) = 0$ ; therefore,

$$\begin{aligned}
 L_1^*(C, A) &= \min_{Y_1} [C' C + \delta Y_1' Y_1 + L_0^*(AC + Y_1, A)] \\
 &= C' C = C' I C
 \end{aligned}$$

(where  $I$  is the identity matrix) which occurs when  $Y_1 = 0$ . This is consistent with physical considerations. Continuing in the same manner,

$$\begin{aligned}
 L_2^*(C, A) &= \min_{Y_1} [C' C + \delta Y_1' Y_1 + L_1^*(AC + Y_1, A)] \\
 &= \min_{Y_1} [C' C + \delta Y_1' Y_1 + (AC + Y_1)'(AC + Y_1)] \\
 &= \min_{Y_1} [C'(I + A'A)C + 2C'A'Y_1 + Y_1'Y_1] \\
 &= C' [I + A'(I + \delta^{-1})A] C
 \end{aligned}$$

which occurs when  $Y_1 = -(\delta + 1)^{-1} AC$ . This suggests Theorem 2.

### Theorem 2.

$$L_N^*(C, A) = C' R_N C \tag{6}$$

$$R_1 = I, \quad R_{N+1} = I + A' [R_N^{-1} + \delta^{-1} I] A, \quad N = 1, 2, \dots$$

*Proof:* To establish the theorem rigorously, we proceed by induction.

It was proved for  $N = 1, 2$ , so let the induction hypothesis be that the theorem is true for  $N = K$ , and show that it is true for  $N = K + 1$ . From Eq. 5

$$L_{K+1}^*(C, A) = \min_{Y_1} [C' C + \delta Y_1' Y_1 + L_K^*(AC + Y_1, A)]$$

By hypothesis,

$$L_K^*(AC + Y_1, A) = (AC + Y_1)' R_K (AC + Y_1)$$

Substitution yields

$$L_{K+1}^*(C, A) = \min_{Y_1} [C'(I + A'R_K A) C + 2Y_1' R_K AC + Y_1'(R_K + \delta I) Y_1]$$

Each term in the expression to be minimized is a scalar, and moreover, the first term  $C'(I + A'R_K A) C$  is independent of  $Y_1$ . Accordingly, define

$$F(Y_1) = 2Y_1' R_K AC + Y_1'(R_K + \delta I) Y_1$$

which maps the vectors  $Y_1$  into the reals, and

$$\frac{dF(Y_1)}{dY_1} = \left[ \frac{\partial F(Y_1)}{\partial y_{11}}, \frac{\partial F(Y_1)}{\partial y_{21}}, \dots, \frac{\partial F(Y_1)}{\partial y_{p1}} \right]$$

$$Y_1 = (y_{11}, y_{21}, \dots, y_{p1})'$$

The minimum of  $F(Y_1)$  will occur ( $R_K$  is positive definite) when  $Y_1 = Y_1^* = -(\delta I + R_K)^{-1} R_K AC$ , which will yield the desired result.

The dynamic programming approach used to obtain the form of  $L_N^*$  in Theorem 2 will also yield the optimal policy,  $Y^*$ . The proof of Theorem 2 already gave us the first vector in the optimal policy,

$$Y_{1,N}^*(C) = -(\delta I + R_N)^{-1} R_N AC \quad (7)$$

Since  $X_N$  depends on  $Y_1, Y_2, \dots, Y_{N-1}$ , but not  $Y_N$ , it is obvious that  $Y_{N,N}^*(C) = 0$ . Let us denote the  $i$ th vector in the optimal policy for the  $N$ -stage process (Eq. 4) as  $Y_{i,N}^*(C)$ . Then we can state Theorem 3.

**Theorem 3.**

$$Y_{i,N}^*(C) = Y_{1,N-i+1}^*(X_i^*), \quad i = 1, 2, \dots, N \quad (8)$$

where  $X_i^*$  denotes the state of  $S$  on the  $i$ th step under optimal policy.

To determine  $X_i^*$ , we utilize Theorem 3 recursively and obtain Theorem 4.

**Theorem 4.**

$$X_i^* = \begin{cases} \delta(\delta I + R_{N-i+2})^{-1} A X_{i-1}^*, & i \geq 2 \\ C, & i = 1 \end{cases} \quad (9)$$

#### IV. ASYMPTOTIC THEORY

The natural question at this point is that of the convergence of the loss function under optimal policy. For the remainder of this Section, the only policy used will be the optimal policy.

**Theorem 5.**

$$L_N^*(C, A) \rightarrow L_\infty^*(C, A) < \infty \text{ as } N \rightarrow \infty$$

*Proof.*  $L_N^*(C, A)$  is uniformly bounded. To see this, consider the policy  $Y$ :  $Y_1 = -AC$ ,  $Y_i = 0$ ,  $i = 2, 3, \dots, N$ , in which  $Y$  forces  $S$  to the zero-state on the second step and to remain there afterwards. This policy has loss function

$$\begin{aligned} L_N(C, A, Y) &= \sum_{n=1}^N X_n' X_n + \delta \sum_{n=1}^N Y_n' Y_n \\ &= C' C + \delta (AC)' (AC) = C' (I + \delta A' A) C \end{aligned}$$

which is independent of  $N$ . Since  $L_N^*(C, A)$  is optimal,

$$L_N^*(C, A) \leq L_N(C, A, Y) = C' (I + \delta A' A) C$$

for all  $N$ . This establishes uniform boundedness.

Now it will be demonstrated that  $L_N^*(C, A)$  is a monotonic, non-decreasing function of  $N$ . Since  $Y_{N+1, N+1}^* = 0$ , consider this policy for an  $N$ -stage process; i.e.,

$$Y: Y_{1, N+1}^*, Y_{2, N+1}^*, \dots, Y_{N, N+1}^*$$

The next statement readily follows:

$$L_N^*(C, A) \leq L_N(C, A, Y) = L_{N+1}^*(C, A),$$

The proof of Theorem 5 simply shows that the sequence  $L_n^*(C, A)$  always converges; it is not constructive in that it does not yield the value of the limit or determine the rate of convergence. A partial resolution of these issues will now be attempted.

Before we proceed further, note that the forms for  $L_N^*(C, A)$ ,  $X_{i,N}^*$  and  $Y_{i,N}^*$  are dependent on  $R_N$ . This means that most of the asymptotic theory will depend on the characterization of  $\lim R_N$  as  $N \rightarrow \infty$ . This is rather unfortunate in the sense that the matrix algorithm defining  $R_N$  is highly nonlinear. Hence, some thought will be given to the analysis of Eq. 6.

Assume a solution in the form  $R_N = U_N V_N^{-1}$ . Then

$$\begin{aligned} R_{N+1} &= U_{N+1} V_{N+1}^{-1} = I + A' [V_N U_N^{-1} + \delta^{-1} I]^{-1} A \\ &= I + A' U_N [V_N + \frac{1}{\delta} U_N]^{-1} A. \end{aligned} \quad (10)$$

Assuming  $A$  to be non-singular,

$$U_{N+1} V_{N+1}^{-1} A^{-1} = \begin{bmatrix} A^{-1} & V_N + \frac{1}{\delta} U_N + A' U_N \end{bmatrix} \begin{bmatrix} V_N + \frac{1}{\delta} U_N \end{bmatrix}^{-1}$$

Letting

$$AU_{N+1} = V_N + \frac{1}{\delta} U_N + AA' U_N \quad (11a)$$

$$AV_{N+1} = \frac{1}{\delta} U_N + V_N \quad (11b)$$

We succeed in writing Eq. 6 as two simultaneous, linear, matrix difference equations with constant coefficients. Continuing in this form, from Eq. 11a,

$$\mathbf{A}\mathbf{U}_{N+2} = \mathbf{V}_{N+1} + \frac{1}{\delta} \mathbf{U}_{N+1} + \mathbf{A}\mathbf{A}'\mathbf{U}_{N+1}$$

$$= \mathbf{A}^{-1} [\beta \mathbf{U}_N + \mathbf{V}_N] + \beta \mathbf{U}_{N+1} + \mathbf{A}\mathbf{A}'\mathbf{U}_{N+1}, \quad \beta = \frac{1}{\delta}$$

$$= (\beta \mathbf{I} + \mathbf{A}\mathbf{A}') \mathbf{U}_{N+1} + \beta \mathbf{A}^{-1} \mathbf{U}_N + [\mathbf{U}_{N+1} - \mathbf{A}^{-1} (\beta \mathbf{I} + \mathbf{A}\mathbf{A}') \mathbf{U}_N]$$

$$= [(1 + \beta) \mathbf{I} + \mathbf{A}\mathbf{A}'] \mathbf{U}_{N+1} - \mathbf{A}' \mathbf{U}_N$$

Therefore,

$$\mathbf{U}_{N+2} = [(1 + \beta) \mathbf{A}^{-1} + \mathbf{A}'] \mathbf{U}_{N+1} - \mathbf{A}^{-1} \mathbf{A}' \mathbf{U}_N$$

which is a linear matrix difference equation of the second order with constant coefficients. Note that  $\mathbf{U}_1 = \mathbf{V}_1$ , which is arbitrary. To simplify the algebra, let  $\mathbf{B} = \mathbf{A}' + (1 + \beta) \mathbf{A}^{-1}$ ,  $\mathbf{D} = -\mathbf{A}^{-1} \mathbf{A}'$ , and obtain

$$\mathbf{U}_{N+2} = \mathbf{B}\mathbf{U}_{N+1} + \mathbf{D}\mathbf{U}_N \quad (12)$$

Assume a solution in the form  $\mathbf{U}_N = Z^N \mathbf{F}$ , where  $Z$  is a scalar and  $\mathbf{F}$  a constant  $p \times p$  matrix. Substituting into Eq. 12 and clearing,

$$(Z^2 \mathbf{I} - Z\mathbf{B} - \mathbf{D}) \mathbf{F} = \mathbf{0} \quad (13)$$

In order that non-trivial matrices  $\mathbf{F}$  exist which satisfy Eq. 13,  $Z$  must satisfy the determinantal, or characteristic, equation

$$|Z^2 \mathbf{I} - Z\mathbf{B} - \mathbf{D}| = 0 \quad (14)$$

This characteristic equation is of degree  $2p$  in  $Z$ , and every root of Eq. 14 will yield an  $\mathbf{F}$  so that  $Z^N \mathbf{F}$  satisfies Eq. 13, which is a necessary condition that it satisfy Eq. 12.

The remainder of the technique will be illustrated by examples, and a lemma will be stated for the case  $p = 2$ .

*Lemma 1.* If  $p = 2$ , and  $A$  is non-singular, then the characteristic equation (14) becomes

$$Z^4 - (\text{trace } B) Z^3 + (|B| - \text{trace } D) Z^2 - (\text{trace } B) Z + 1 = 0 \quad (15)$$

Notice that in this case, if  $Z$  is a root of Eq. 15, then  $1/Z$  is a root of 15 also.

*Example 1.* Let  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ,  $\delta = 1$ . Then  $B = 3I$  and  $D = -I$ , and Eq. 14 becomes

$$Z^4 - 6Z^3 + 11Z^2 - 6Z + 1 = (Z^2 - 3Z + 1)^2 = 0$$

Solving the quadratic, we get two distinct real roots, each of multiplicity two, namely  $Z_1 = (3 + \sqrt{5})/2$  and  $Z_2 = (3 - \sqrt{5})/2$ . Write  $U_N = Z_1^N F_1 + Z_2^N F_2$ , and notice that  $U_N \sim Z_1^N F_1$  since  $0 < Z_2 < 1 < Z_1$ . To obtain  $V_N$ , we go back to Eq. 11a:

$$\begin{aligned} V_N &= U_{N+1} - (\beta I + I) U_N \\ &= Z_1^{N+1} F_1 + Z_2^{N+1} F_2 - 2Z_1^N F_1 - 2Z_2^N F_2 \\ &= Z_1^N \left( \frac{\sqrt{5} - 1}{2} \right) F_1 - Z_2^N \left( \frac{\sqrt{5} + 1}{2} \right) F_2 \\ V_N &\sim Z_1^N \left( \frac{\sqrt{5} - 1}{2} \right) F_1 \end{aligned}$$

and hence,

$$V_N^{-1} \sim Z_1^{-N} \left( \frac{2}{\sqrt{5} - 1} \right) F_1^{-1}$$

Therefore,

$$U_N V_N^{-1} \sim \frac{2}{\sqrt{5} - 1} I$$

which one can verify as the correct answer.

Theorem 5 states that we have convergence for all  $A$  and  $\delta > 0$ . Thus, Example 2 is designed to show what can be done when  $A$  is singular.

*Example 2.* Let  $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ ,  $\delta = 0.5$ . Since  $A$  is singular, multiply Eq. 13 on the left by  $A$  and use  $|Z^2 A - [(1 + \beta) I + AA'] Z + A'| = 0$  as the characteristic equation. This yields  $|Z^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 2 \\ 2 & 5 \end{bmatrix} Z + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}| = 0$ , whose roots are  $Z_1 = (7 + \sqrt{33})/4$ ,  $Z_2 = (7 - \sqrt{33})/4$ , and  $Z_3 = 0$ . Notice that a singular  $A$  may yield a characteristic polynomial of degree less than  $2p$ . Write  $U_N = Z_1^N F_1 + Z_2^N F_2$ , ( $Z_3 = 0$ ); then  $U_N \sim Z_1^N F_1$ . Again,  $V_N$  can be computed from Eq. 11a:

$$\begin{aligned} V_N &= Z_1^{N+1} A F_1 + Z_2^{N+1} A F_2 - (\beta I + AA')(Z_1^N F_1 + Z_2^N F_2) \\ &= Z_1^N [Z_1 A - \beta I - AA'] F_1 + Z_2^N [Z_2 A - \beta I - AA'] F_2 \end{aligned}$$

$$V_N^{-1} \sim Z_1^{-N} F_1^{-1} [Z_1 A - \beta I - AA']^{-1}$$

Therefore,

$$R_N \sim [Z_1 A - \beta I - AA']^{-1}$$

Completing the calculations we find that

$$R = \frac{1}{4(\sqrt{33} - 5)} \begin{bmatrix} 9 - \sqrt{33}, & \sqrt{33} - 1 \\ \sqrt{33} - 1, & 9 - \sqrt{33} \end{bmatrix}$$

The examples also illustrate Theorem 6.

**Theorem 6.**  $R_N$  converges exponentially to  $R$ .

A direct solution may be obtained for  $R$  when  $A = I$ . In this case, the matrix equation to be solved is

$$R = I + A' \left[ R^{-1} + \frac{1}{\delta} I \right]^{-1} A$$

$$R = I + \left[ R^{-1} + \frac{1}{\delta} I \right]^{-1}$$

$$R \left[ R^{-1} + \frac{1}{\delta} I \right] = R^{-1} + \frac{1}{\delta} I + I$$

$$I + \frac{1}{\delta} R = R^{-1} + \left( 1 + \frac{1}{\delta} \right) I$$

Multiplying both sides by  $\delta R$  and collecting terms,

$$R^2 - R - \delta I = 0$$

Assume a solution in the form  $R = aI$ , where  $a$  is a real number then

$$[a^2 - a - \delta] I = 0$$

or

$$a = \frac{1 \pm \sqrt{1 + 4\delta}}{2}$$

Since  $R$  is positive definite, only the positive sign can be used, and we obtain

$$R = \frac{1 + \sqrt{1 + 4\delta}}{2} I$$

If  $p = 2$  and  $\delta = 1$ , then

$$R = \frac{1 + \sqrt{5}}{2} I,$$

which agrees with results obtained.

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